Second Fundamental Theorem of Calculus

The second fundamental theorem is a precise way of showing how differentiation “undoes” integration. The basic theorem reads as follows:

**Second Fundamental Theorem of Calculus:** If $f(x)$ is continuous on an open interval $I$ containing $a$, then, for every $x$ in the interval,

$$\frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x).$$

**Example 1:** To see how the theorem is applied let $f(t) = \tan(t^3)$. Then we have:

$$y = \int_a^x \tan(t^3) \, dt.$$

According to the second fundamental theorem, upon taking the derivative of $y$ with respect to $x$, we need only replace $t$ with $x$ in the integrand and remove the integral giving:

$$\frac{dy}{dx} = \tan(x^3).$$

Often times a more sophisticated version of the theorem is needed. This more powerful version reads:

**General Second Fundamental Theorem of Calculus:** If $f$ is continuous on an open interval $I$ containing $a$ and $u(x)$ is a differentiable function of $x$ when $u(x)$ belongs to that interval, then, for every $x$ such that $u(x)$ is in the interval $I$,

$$\frac{d}{dx} \left[ \int_a^{u(x)} f(t) \, dt \right] = f(u(x)) \cdot \frac{du}{dx}.$$

This result follows directly from an application of the chain rule along with the second fundamental theorem.

**Example 2:** To see how the theorem is applied let $f(t) = \tan(t^3)$ and $u(x) = \sin(x)$. Then we have:

$$y = \int_a^{\sin(x)} \tan(t^3) \, dt.$$

According to the general second fundamental theorem, upon taking the derivative of $y$ with respect to $x$, we need only replace $t$ with $u(x) = \sin(x)$ in the integrand, multiply by $\frac{du}{dx} = \cos(x)$, and remove the integral giving:

$$\frac{dy}{dx} = \tan(\sin^3(x)) \cdot \cos(x).$$

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